UDC 539.3

APPROXIMATE METHOD OF SOLUTION OF DYNAMIC PROBLEMS FOR LINEAR VISCOELASTIC MEDIA

PMM Vol. 43, No. 1, 1979, pp. 133-137

I. G. FILIPPOV
(Moscow)

(Received February, 6, 1978)

An approximate method of solution of dynamic problems for linear viscoelastic media exhibiting an instantaneous elasticity, is given. The method yields exact results for media which satisfy the Maxwell model. In the past the investigations were focused mainly on the simplest problems with the kernels of viscoelastic operators assuming particular forms, especially for the Maxwell body or in the neighborhood of a viscoelastic wavefront [1 — 3], and on the asymptotic solutions using the method of steepest descent [4].

1. Formulation of the problem and a method of approximate solution. Linear viscoelastic bodies exhibiting instantaneous elasticity are described by two linear, time-dependent Boltzmann-type operators $L(\zeta)$ and $M(\zeta)$

$$L(\zeta) = \lambda\left[\zeta(t) - \int_0^t f_1(t-\xi)\zeta(\xi)\,d\xi\right] \tag{1.1}$$

$$M(\zeta) = \mu\left[\zeta(t) - \int_0^t f_2(t-\xi)\zeta(\xi)\,d\xi\right]$$

where $f_1(t)$ and $f_2(t)$ are kernels of the viscoelastic operators which are approximated more conveniently by expressing them in the form of a sum of exponential terms, i. e. [5]

$$f_j(t) = \sum_{m=1}^n \frac{\gamma_{mj}}{\tau_m}\exp\left(-\frac{t}{\tau_m}\right), \quad \sum_{m=1}^n \gamma_{mj} = 1 \tag{1.2}$$

λ and μ are elastic constants, γ_{mj} are the viscosity parameters and τ_m denote the relaxation times.

We shall limit ourselves, for simplicity, to considering two-dimensional plane problems, and reduce the equations of motion of a linear, isotropic, viscoelastic medium, to the form

$$L_j(\Delta u_j) = \frac{1}{a_j^2}\frac{\partial^2 u_j}{\partial t^2}, \quad L_j(\Delta v_j) = \frac{1}{a_j^2}\frac{\partial^2 v_j}{\partial t^2}, \quad j = 1, 2 \tag{1.3}$$

$$L_1(\zeta) = \frac{1}{\lambda + 2\mu}[L(\zeta) + 2M(\zeta)], \quad L_2(\zeta) = \frac{1}{\mu}M(\zeta)$$

where the functions u_j and v_j must satisfy the following additional relations:

$$\frac{\partial u_1}{\partial y} = \frac{\partial v_1}{\partial x}, \quad \frac{\partial u_2}{\partial x} = -\frac{\partial v_2}{\partial y}, \quad a_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad a_2^2 = \frac{\mu}{\rho}$$

When the Poisson's ratio is constant, which is true to within a large degree of accuracy, the kernels $f_1(t)$ and $f_2(t)$ become equal to each other.

To elucidate the approximate method, we consider an arbitrary integro-differential equation

$$L(\Delta u) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

of the type (1.3), and apply to it a Laplace transform in t. In the case of zero initial conditions, which is not essential in the present approach, we obtain the following equation for $u_0(x, y, p)$:

$$\Delta u_0 = \frac{Q^2(p)}{c^2}u_0, \quad Q^2(p) = p^2[1 - f_0(p)]^{-1}$$

$$f_0(p) = \int_0^\infty f(t)\exp(-pt)\,dt$$

where c denotes any one of the velocities a_j. The quantity $Q^2(p)$ is written for the kernel $f(t)$ of the form (1.2) in the following manner:

$$Q^2(p) = p^2 + c_1 p - c_2 + c_2 Q_1(p) \tag{1.4}$$

$$c_1 = \sum_{m=1}^{n}\frac{\gamma_m}{\tau_m}, \quad c_2 = \sum_{m=1}^{n}\sum_{l=1+m}^{n}\gamma_m\gamma_l\left(\frac{1}{\tau_m} - \frac{1}{\tau_l}\right)^2$$

$$|Q_1(p)| \leqslant \left|\frac{1}{1 + p\tau_1}\right|$$

In the case of a Maxwellian material we have a single relaxation time, and the quantity $Q^2(p)$ simplifies to

$$Q^2(p) = p^2 + p/\tau_1 \tag{1.5}$$

Let us make the following assumption. Since in many dynamic problems for linear viscoelastic media the time taken by the wave processes is much shorter than the shortest relaxation time τ_1, it follows that for the time intervals in question, which is t, we can neglect the last term in (1.4) (for a Maxwellian body this term will be identically equal to zero).

2. One-dimensional plane wave in a viscoelastic medium. Let us consider the simplest problem of propagation of a plane wave in a half-space $y > 0$, the wave generated by a normal pressure applied at the boundary $y = 0$ of this half-space. The problem reduces to that of solving the equation

$$L_1\left(\frac{\partial^2 v}{\partial y^2}\right) = \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}, \quad a_1 = a$$

for the displacement v, satisfying the following boundary and initial conditions:

$$\sigma_{yy} = -F(t) \quad (y = 0), \quad v = 0, \quad (y = \infty)$$

$$v = \partial v/\partial t = 0 \quad (t = 0)$$

Applying now the Laplace transformation in t, we obtain a problem the solution of which has the form

$$v_0(y, p) = \frac{Q(p)}{\rho a p} F_0(p) \exp\left[-Q(p)\frac{y}{a}\right] \tag{2.1}$$

Solving (2.1) for p, in which case the quantity $Q^2(p)$ is given by the formula(1.4) with the last term omitted, we obtain the following expressions for the displacement v and stress σ_{yy} :

$$v(y,t) = \frac{1}{\rho a}\int_{y/a}^{t} F(t-\xi)\left[\exp\left(-\frac{c_1\xi}{2}\right) I_0\left(c_0\sqrt{\xi^2 - \frac{y^2}{a^2}}\right) + c_0\int_{y/a}^{\xi} \exp\left(-\frac{c_1\eta}{2}\right) I_0\left(c_0\sqrt{\eta^2 - \frac{y^2}{a^2}}\right) d\eta\right] d\xi$$

$$c_0 = \sqrt{\frac{c_1^2}{4} + c_2}$$

$$\sigma_{yy}(y,t) = -\exp\left(-\frac{c_1 y}{2a}\right) F\left(t - \frac{y}{a}\right) + \frac{c_0 y}{a}\int_{y/a}^{t} F(t-\xi) \times \exp\left(-\frac{c_1\xi}{2}\right)\left(\xi^2 - \frac{y^2}{a^2}\right)^{-1/2} I_1\left(c_0\sqrt{\xi^2 - \frac{y^2}{a^2}}\right) d\xi$$

For the Maxwellian body this yields an exact solution which was already constructed in [1] using the correspondence principle, while in the case of an elastic body where $f(t) = 0$, we obtain a known solution for the propagation of a plane wave through an elastic medium. An approximate relation connecting the stress with the velocity of the particles behind the wave front is also easily obtained

$$\sigma_{yy}(y,t) = -\rho a\left\{\frac{dv}{dt} + \int_{y/a}^{t} \frac{dv}{\partial\xi}\frac{\partial}{\partial\xi}\left[\exp\left(-\frac{c_1\xi}{2}\right) I_0(c_0\xi)\right] d\xi\right\}$$

and it represents a well known relation for a plane, elastic wave.

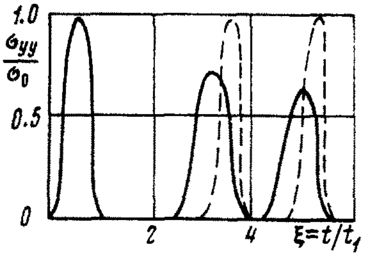

Fig. 1

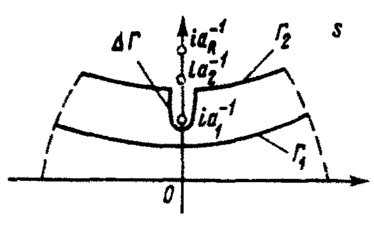

Fig. 2

Fig. 1 depicts the dependence of the magnitude of the stress σ_{yy} at two points and the dependencies on t , for the following initial values of the problem:

$$F(t) = \sigma_0 \sin^2(\pi t/t_1) \quad (0 \leqslant t \leqslant t_1), \quad n = 3, \quad \gamma_1 = 0.6$$
$$\gamma_2 = 0.3, \quad \gamma_3 = 0.1, \quad \tau_1 = 10^5\ \mu\text{s}, \quad \tau_2 = 10^7\ \mu\text{s},$$
$$\tau_3 = 10^7\ \mu\text{s}$$

with the dashed line corresponding to the elastic problem. Other one-dimensional problems of propagation of plane waves can be solved in an analogous manner.

3. **Approximate solution of the Lamb's problem.** The problem is reduced to that of solving the equations (1.3) for a half-plane $y < 0$, uner the following boundary and initial conditions:

$$\sigma_{yy} = \sigma_0\delta(x)\delta(t), \quad \sigma_{xy} = \sigma_1\delta(x)\delta(t) \quad (y = 0)$$
$$u_j = v_j = 0 \quad (\sqrt{x^2 + y^2} = \infty)$$
$$u_j = v_j = \partial u_j/\partial t = \partial v_j/\partial t = 0 \quad (t = 0)$$

Assuming that the Poisson's ratio is constant, we apply to the problem the Laplace transform in t and the Fourier transform in x. Then for the functions

$$\bar{u}_{j0} = \int_{-\infty}^{\infty} dx \int_0^{\infty} u_j \exp(i\omega x - pt)\, dt$$
$$\bar{v}_{j0} = \int_{-\infty}^{\infty} dx \int_0^{\infty} v_j \exp(i\omega x - pt)\, dt$$

we obtain

$$\bar{u}_{10} = A_1 \exp[yR_1(p)], \quad \bar{v}_{20} = A_2 \exp[yR_2(p)] \tag{3.1}$$
$$\bar{u}_{20} = -i\omega^{-1}R_2(p)\,\bar{v}_{20}, \quad \bar{v}_{10} = i\omega^{-1}R_1(p)\,\bar{u}_{10}$$
$$R_j(p) = [\omega^2 + Q^2(p)/a_j^2]^{1/2}, \quad A_j = \Delta_j/\Delta, \quad j = 1, 2$$
$$\Delta = [(2\omega^2a_2^2 + Q^2(p))^2 - 4\omega^2a_2^4R_1(p)\,R_2(p)]$$
$$\Delta_1 = -\frac{\omega Q^2(p)}{\rho p^2 a_2^2}[i\sigma_0(2\omega^2a_2^2 + Q^2(p)) + 2\omega^2a_2^2\sigma_1R_2(p)]$$
$$\Delta_2 = -\frac{\omega Q^2(p)}{\rho p^2 a_2^2}[2\omega^2a_2^2\sigma_0R_1(p) - i\sigma_1(2\omega^2a_2^2 + Q^2(p))]$$

Let us inspect the values of the stresses. Using the solution in the form (3.1), we obtain the following expressions for the Laplace and Fourier transforms of the stresses σ_{ij}^0:

$$\sigma_{ij}^{(0)} = Q(p)\sum_{k=1}^{2}\int_{-\infty}^{\infty} P_{ij}^{(k)}(s)\exp[Q(p)(yR_{j0}(s) - isx)]\, ds \tag{3.2}$$
$$P_{xx}^{(1)} = -i\frac{\Delta_{10}}{\Delta_0}[a_1^2s^2 - (a_1^2 - 2a_2^2)R_{10}^2(s)], \quad P_{xx}^{(2)} = -P_{yy}^{(2)}$$
$$P_{xx}^{(2)} = -2s\frac{\Delta_{20}}{\Delta_0}R_{20}, \quad P_{yy}^{(1)} = -i\frac{\Delta_{10}}{\Delta_0}[s^2(a_1^2 - 2a_2^2) -$$
$$a_1^2R_{10}^2(s)]$$
$$P_{xy}^{(1)} = 2sa_2^2\frac{\Delta_{10}}{\Delta_0}R_{10}(s), \quad P_{xy}^{(2)} = -i\frac{\Delta_{20}}{\Delta_0}R_0(s)$$
$$R_{j0}(s) = \sqrt{s^2 + a_j^{-2}}, \quad R_0(s) = (1 + 2s^2a_2^2)$$
$$\Delta_0 = [R_0^2(s) - 4s^2a_2^4R_{10}(s)\,R_{20}(s)]$$
$$\Delta_{10} = -[i\sigma_0R_0(s) + 2sa_2^2\sigma_1R_{20}(s)]$$
$$\Delta_{20} = -[2sa_2^2\sigma_0R_{10}(s) - i\sigma_1R_0(s)]$$

To analyze the expressions (3.2), we pass to the complex s -plane and replace the integration with respect to s by integration along the contours Γ_1, Γ_2 and $\Delta\Gamma$, as shown in Fig. 2. The branch points and poles of the integrand functions are given by

$$\operatorname{Re} s_1 = 0, \quad \operatorname{Re} s_2 = 0, \quad \operatorname{Im} s_1 = a_1^{-1}, \quad \operatorname{Im} s_2 = a_2^{-1}$$
$$\operatorname{Re} s_R = 0, \quad \operatorname{Im} s_R = a_R^{-1}, \quad \Delta_0(s)|_{s=s_R} = 0$$

Using the method given in [6] and computing the quadratures in (3.2) along the contours Γ_1, Γ_2 and $\Delta\Gamma$ with the help of the Jordan lemma and theory of residues, we can obtain the expressions for the stresses $\sigma_{ij}^{(0)}$ in the half-plane $y \leqslant 0$. For example, we obtain the following expression for $\sigma_{xx}^{(0)}$ at the free boundary $y = 0$, for $\sigma_1 = 0$:

$$\sigma_{xx}^{(0)} = Q(p)\{[P_{xx}^{(1)} + P_{xx}^{(2)}](s - ia_R^{-1}) \exp[i|x|sQ(p)]\}_{s=ia_R^{-1}} + \tag{3.3}$$
$$Q(p)\int_{a_2^{-1}}^{a_1^{-1}} \sigma_0 \left[\frac{F_1(i\eta)}{\Delta_0(i\eta)} + \frac{F_2(i\eta)}{\bar{\Delta}_0(i\eta)}\right] \exp[-\eta|x|Q(p)]\, d\eta$$
$$\bar{\Delta}_0(s) = [R_0^2(s) + 4s^2a_2^4R_{10}(s)R_{20}(s)]$$
$$F_{1,2}(s) = 4s^2R_{10}(s)R_{20}(s) \mp (2s^2 + a_2^{-2})[2s^2 - a_2^{-2}(1 - 2a_2^2/a_1^2)]$$

The above expression was derived incorrectly in a similarly formulated problem in [7].

In order to invert the expression (3.3) with respect to p, we must invert

$$Q(p)\exp[-\alpha Q(p)]$$

and this is easily done for the quantity $Q^2(p)$ defined by (1.4). We obtain

$$\frac{1}{2\pi i}\int_{\alpha_0-i\infty}^{\alpha_0+i\infty} Q(p)\exp[-\alpha Q(p) + pt]\, dp =$$
$$\exp(-{}^1/_2 c_1 t)\{\delta'(t-\alpha) I_0(c_0\sqrt{t^2-\alpha^2}) -$$
$$2\delta(t-\alpha)(t^2-\alpha^2)^{-1/2} I_1(c_0\sqrt{t^2-\alpha^2}) +$$
$$c_0[[(t^2-\alpha^2)^{-1/2} - {}^1/_2\alpha^2(t^2-\alpha^2)^{-3/2}] I_1(c_0\sqrt{t^2-\alpha^2}) +$$
$$c_0^2\alpha^2(t^2-\alpha^2)^{-1/2} I_1'(c_0\sqrt{t^2-\alpha^2})] H(t-\alpha)\}$$

For a viscoelastic half-plane made of material conforming to the Maxwell's model, the formula (3.3) yields an exact expression for $\sigma_{xx}^{(0)}$ at the boundary $y = 0$.

REFERENCES

1. B l a n d, D. Theory of Linear Viscoelasticity. Moscow, "Mir", 1965.
2. C h r i s t e n s e n, R. M., Viscoelasticity, an Introduction. Academic Press, 1971.
3. G a l i n, L. A. and S h m a t k o v a, A. A. Motion of a rigid stamp on the boundary of a viscoelastic half-plane. PMM. Vol. 32, No. 3, 1968.
4. B r u k, S. Z. Lamb problem for a viscoelastic half-plane, Izv. Akad. Nauk SSSR, MTT, No. 3, 1972.

5. F i l i p p o v, I. G. Dynamic problems of the linear theory of viscoelasticity. In the book: Selected Problems of Applied Mechanics, Moscow, VINITI, 1974.

6. C a g n i a r d, L. Reflection and refraction of progressive seismic waves, New York, McGraw —Hill, 1962.

7. C h w a l c z y k, F., R a f a, J., W l o d a r c z y k, E. Propagation of two-dimensional nonstationary stress waves in a semiinfinite viscoelastic body, produced by a normal load moving over surface with subsonic velocity. Proc. Vibrat. Problems, Vol. 13, No. 3, 1972.

Translated by L. K.